Analysis for bifurcation phenomena of nonlinear vibrations (Numerical Solution of Partial Differential Equations and Related Topics II)

Kawanago, Tadashi

数理解析研究所講究録
1198巻 2001年 13-20

Analysis for bifurcation phenomena of nonlinear vibrations

東京工業大学 川中子 正 (Tadashi Kawanago)

Department of Mathematics, Tokyo Institute of Technology
e-mail address: tadashi@math.titech.ac.jp

0. Introduction

Combining some analytical methods and some numerical methods, we are able to analyse rigorously some (symmetry-breaking) bifurcation phenomena of periodic solutions of semilinear (hyperbolic and parabolic) partial differential equations (PDEs) describing some dissipative forced vibration systems. We adopt some methods in [O], where Oishi treats the Duffing equation. The outline of analysis is the following: In order to apply a bifurcation theorem ([WS], [K1]), we use some numerical method and find an approximate solution (= bifurcation point) with high accuracy of the extended system. Then, we show the unique existence of a real solution in a neighborhood of the approximate solution by using the convergence theorem of the Newton's method ([K1], [O]). The most difficult part in the analysis for concrete problems is to check the conditions of the convergence theorem of the Newton's method. More concretely, we have the following two difficult points:

(I) to construct an highly accurate approximate solution of bifurcation point

(II) to estimate the operator norm of the inverse of a linearized operator at bifurcation point

In this article we mainly intend to explain how to treat (I) and (II). For (I) we propose some methods by using the least square methods (see Section 2). For (II) we present some methods in which we approximate linearlized operators by some operators with simpler structure (see Section 3). In view of the form, the author calls the latter operators '*diagonal operators*'. When we treat PDEs, the highly accurate approximate solution we need have in general very long terms. A lot of difficulties arise from such long length of approximate solutions. Even for one-dimensional vibration problem approximate solution we treat by the use of the spectral methods have more than one hundred terms. It seems that for higher-dimensional problems we need approximate solution with much more terms.

1. Our main example

Let $f(\lambda, u) := u_{tt} - c^2 u_{xx} + \mu u_t + u^3 - \lambda \sin x \cos t$. Here, $c, \mu > 0$ are constants and $\lambda > 0$ is a parameter. We consider the bifurcation phenomena of periodic solutions for the following semilinear wave equation:

$$\text{(P)} \quad \begin{cases} f(\lambda, u) = 0 & \text{in} \quad (0, \pi) \times \mathbf{R}^+, \\ u(0,t) = u(\pi, t) = 0 & \text{for} \quad t \geq 0. \end{cases}$$

This has some deep relations to the ordinalry differential equation called the Duffing equation. By some numerical simulations we can observe rich bifurcation phenomena (such as the existence of turning points, symmetry-breaking bifurcations, chaos) for our problem (P) and the Duffing equation. It is difficult to analyse mathematically such phenomena only by the use of pure analytical methods, except for some problems with very restricted artificial forcing terms ([M], [Ko]). Oishi explains in [O] that we can analyse rigorously some bifurcation phenomena of the Duffing equations with natural forcing terms.

The equation in our problem (P) has some symmetry. Let S be the transformation defined by

$$S: \ u(x,t) \longmapsto -u(x, t+\pi).$$

Then we have $f(\lambda, Su) = Sf(\lambda, u)$. The symmetric periodic solution (resp. the asymmetric periodic solution) is a solution satisfying $Su = u$ (resp. $Su = u$).

Let $c = 1$, $\mu = 0.1$. Let us move the value of λ gradually larger from 0. Then we can observe that a branch of asymmetric 2π-periodic solutions bifurcates from a branch of symmetric 2π-periodic solutions at a certain value $\lambda = \lambda_0 \in (2.7, 2.8)$ by some numerical simulations. We can prove this and the following holds:

Proposition 1.1. ([K2]) *There exists a symmetric branch $\{(\lambda, u_\lambda)\}_\lambda$ of 2π-periodic soluitons of (P) which has a symmetry-breaking pitchfork bifurcation point (λ_0, u_0) such that*

$$|\lambda_0 - \Lambda_0|^2 + \|u_0 - U_0\,;\, H^1(D)\|^2 \leq (0.014)^2.$$

Here, $D := (0,\pi) \times (0, 2\pi)$, $\Lambda_0 := 2.767$ and
$U_0 := 1.675064 \cos t \sin x + \cdots + 6.878066 \times 10^{-6} \sin 21t \sin 21x$ has the form of a finite Fourier expansion consisting of 160 terms. We omit here the complete form of U_0.

In order to prove this result, we use a bifurcation theorem [K1, Theorem 3.1] which is a generalized version of [WS, Thorem 3.1]. We define a closed linear subspace X in $H^1(D)$:

$$X := \{ \sum_{\substack{m\in\mathbf{Z}\\ n\in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx \,;\ \sum_{\substack{m\in\mathbf{Z}\\ n\in 2\mathbf{N}-1}} (m^2+n^2)|a_{mn}|^2 < \infty \}$$

and define the symmetric subspace X_s and the anti-symmetric subspace X_a:

$$X_s := \{ \sum_{\substack{m\in 2\mathbf{Z}-1\\ n\in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx \,;\ \sum_{\substack{m\in 2\mathbf{Z}-1\\ n\in 2\mathbf{N}-1}} (m^2+n^2)|a_{mn}|^2 < \infty \},$$

$$X_a := \{ \sum_{\substack{m\in 2\mathbf{Z}\\ n\in 2\mathbf{N}-1}} a_{mn} e^{imt} \sin nx \,;\ \sum_{\substack{m\in 2\mathbf{Z}\\ n\in 2\mathbf{N}-1}} (m^2+n^2)|a_{mn}|^2 < \infty \}.$$

Then, we have $X = X_s \oplus X_a$. Let $Y := \overline{X}^{L^2(D)}$. We define Y_s and Y_a similaly to the definition of X_s and X_a. Let L_0 be a principal part of f, i.e.

$$L_0 := u_{tt} - c^2 u_{xx} + \mu u_t.$$

The operator $L_0 : X \to Y$ is an *unbounded* operator with $L_0^{-1} \in \mathcal{L}(Y, X)$. The unboundedness of the operator is the special feature commonly in the analysis for PDEs with time variable. For this reason we formulate in [K1] generalize versions of some bifurcation theorems, which is applicable to non-Fréchet differentiable maps.

In view of bifurcation theorem [K1, Theorem 3.1], we should show that (λ_0, u_0) is a simple singular point of $f(\lambda, u)$ and that an extended system:

$$F(\lambda, u, \phi) := \begin{pmatrix} l\phi - 1 \\ f(\lambda, u) \\ D_u f(\lambda, u)\phi \end{pmatrix} = 0$$

has an isolated solution (λ_0, u_0, ϕ_0), where $l \in X^*$ is a functional to normalize ϕ_0 and $F : \mathbf{R} \times X_s \times X_a \to \mathbf{R} \times Y_s \times Y_a$. We should construct an approximate solution of (λ_0, u_0, ϕ_0) and apply the convergence theorem of Newton's method [K1, Theorem 1.1].

2. Construction of approximate solution with high accuracy

In this section we will explain how to construct a highly accurate approximate solutions of partial differential equations. One of advantage of our method below is that we can construct a more highly accurete approximate solution by adding some new high frequency terms provided we notice the accuracy of our approximate solution is not sufficient.

As an example we treat (P) in Section 1. We fix λ and set $L(u) := f(\lambda, u)$ for simplicity. We will construct an approximate 2π- periodic solution with high accuracy. We set the inner product:

$$(g, h) := \int_0^{2\pi} dt \int_0^{\pi} dx\, g(x,t)\, \overline{h(x,t)}.$$

We assume that we already obtain an approximate solution:

$$u^0 = \sum_{(k,l)\in A} a^0_{kl}\, e^{ikt} \sin lx$$

by some methods (such as the finite difference methods or finite element methods). Here, $i = \sqrt{-1}$ and

$$A := \{(k,l) \in \mathbf{N} \times \mathbf{N}\ ;\ |k|, l \leq M\}.$$

First let u_1 be an 'optimized' function of u_0 by the least-squares methods:

$$u^1 = \sum_{(k,l)\in A} a_{kl}\, e^{ikt} \sin lx. \tag{2.0}$$

Here, we determine unknown number a_{kl} such that $(\,L(u^1), L(u^1)\,)$ takes the minimum. Namely, we determine a_{kl} by applying the Newton method with the initial value: $a_{kl} = a^0_{kl}$ to the simultaneous equations

$$\frac{\partial}{\partial a_{kl}}(\,L(u^1), L(u^1)\,) = 0, \quad (k,l) \in A. \tag{2.1}$$

In order to obtain an approximate solution with higher accuracy we set

$$u^2 = u^1 + \sum_{(k,l)\in B-A} \alpha_{kl}\, e^{ikt} \sin lx \tag{2.2}$$

$$\text{with} \quad B := \{(k,l) \in \mathbf{N} \times \mathbf{N}\ ;\ |k|, l \leq N\},\ M < N$$

and we determine unknown number α_{kl} such that $(\,L(u^2), L(u^2)\,)$ takes the minimum. Namely, We make a simultaneous equations for α_{kl} similar to (2.1) and determine α_{kl} by applying the Newton method with the initial value: $\alpha_{kl} = 0$. After the determination of α_{kl} we optimize the part of u^1 in the right-hand side of (2.2). If necessary, we repeat the similar procedure to obtain an approximate solution with higher accuracy.

3. To estimate the norm of inverse operator

First, we prepare two lemmas:

Lemma 3.1. *Let X and Y be Banach spaces. Let $L : X \to Y$ be a (maybe unbounded) linear operator. Assume that there exists $P \in \mathcal{L}(Y, X)$ such that*

$$\begin{cases} LP \in \mathcal{L}(Y, X), \\ \|LP - I\| < 1. \end{cases}$$

Then we have $L^{-1} \in \mathcal{L}(Y, X)$ with estimates:

$$\|L^{-1} - P\| \leq \frac{\|P\|\|LP - I\|}{1 - \|LP - I\|},$$

$$\|L^{-1}\| \leq \frac{\|P\|}{1 - \|LP - I\|}.$$

Lemma 3.2. *Let X and Y be Banach spaces. Let $L : X \to Y$ be a (maybe unbounded) linear operator. Assume that there exists a linear operator $A : X \to Y$ such that*

$$\begin{cases} A^{-1} \in \mathcal{L}(Y, X), \\ LA^{-1} \in \mathcal{L}(Y, Y), \\ L - A \in \mathcal{L}(X, Y) \quad \text{with} \quad \|L - A\|\|A^{-1}\| < 1. \end{cases}$$

Then we have $L^{-1} \in \mathcal{L}(Y, X)$ with estimates:

$$\|L^{-1} - A^{-1}\| \leq \frac{\|L - A\|\|A^{-1}\|^2}{1 - \|L - A\|\|A^{-1}\|},$$

$$\|L^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|L - A\|\|A^{-1}\|}.$$

Let X and Y be Banach spaces with $X \hookrightarrow Y$ being dense. Let $L : X \to Y$ be a (maybe unbounded) linear operator. In this section we will show how to estimate $\|L^{-1}\|$. We will approximate L by the following operator A:

$$A \simeq \begin{pmatrix} A_N & 0 \\ 0 & S \end{pmatrix}, \tag{3.1}$$

where A_N is a finite $N \times N$ matrix and S is an infinite diagonal matrix. The relation (3.1) means that there exists a complete orthorgonal base $\{\phi_k\}_{k=1}^{\infty}$ of Y such that

$\{\phi_k\}_{k=1}^{\infty} \subset X$ and $A_N X_N \subset X_N$ with $X_N := [\phi_1, \cdots, \phi_N]$ (which means the linear subspace generated by $\phi_1, \cdots, \phi_N$) and ϕ_k $(k \geq N+1)$ is an eigenvector of $S : X_r \to \overline{X_r}^Y$ with $X_r := [\phi_{N+1}, \phi_{N+2}, \cdots]$. We set

$$P := A^{-1} = \begin{pmatrix} A_N^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix}, \tag{3.2}$$

Let

$$x = \begin{pmatrix} x_N \\ x_r \end{pmatrix} \in X \quad \text{with} \quad x_N \in X_N \quad \text{and} \quad x_r \in X_r.$$

Then, we have

$$\begin{aligned}(LA^{-1} - I)x &= L \begin{pmatrix} A_N^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} x_N \\ x_r \end{pmatrix} - \begin{pmatrix} x_N \\ x_r \end{pmatrix} \\ &= (LA_N^{-1} - I)x_N + (LS^{-1} - I)x_r\end{aligned}$$

It follows that

$$\begin{aligned}\|(LA^{-1} - I)x\| &\leq \|LA_N^{-1} - I\| \, \|x_N\| + \|LS^{-1} - I\| \, \|x_r\| \\ &\leq (\|LA_N^{-1} - I\|^2 + \|LS^{-1} - I\|^2)^{1/2} \|x\|\end{aligned}$$

Therefore, we obtain that

$$\|LA^{-1} - I\| \leq (\|LA_N^{-1} - I\|^2 + \|LS^{-1} - I\|^2)^{1/2}$$

The following condition is sufficient to apply our Lemma 3.1:

$$\begin{cases} \|LA_N^{-1} - I\| \leq 1/2, \\ \|LS^{-1} - I\| \leq 1/2. \end{cases} \tag{3.3}$$

We remark that our method is actually applicable to problems with higher nonlinearity than our main problem (P). Therefore, we consider, as an example, the following simple problem:

$$L := -\frac{\partial^2}{\partial x^2} + p(x)\frac{\partial}{\partial x} + q(x) \quad \text{with} \quad p, q \in L^\infty(0, \pi).$$

We set

$$X := H_0^1(0, \pi) \quad \text{or} \quad H^2 \cap H_0^1(0, \pi), \quad \text{and} \quad Y := L^2(0, \pi).$$

We will show how to construct an operator A which approximates L . Let $P(x)$ (resp. $Q(x)$) be a finite Fourier Sine expansion (resp. a finite Fourier Cosine expansion) in $(0, \pi)$ approximating $p(x)$ (resp. $q(x)$). We set

$$\tilde{L} := -\frac{\partial^2}{\partial x^2} + P(x)\frac{\partial}{\partial x} + Q(x).$$

Assumption: $\tilde{L}$ sufficiently approximates L, i.e. $\|L - \tilde{L}\|$ is sufficiently small.

We remark that this assumption holds in general for applications including the analysis for our problem (P). In view of Lemma 3.2 we can reduce the estimate of $\|L^{-1}\|$ to that of $\|\tilde{L}^{-1}\|$. Therefore, we assume below without loss of generality that $p(x)$ (resp. $q(x)$) has the form of a finite Fourier Sine expansion (a finite Fourier Cosine expansion). We define $S := -\partial^2/\partial x^2$ and $\phi_n := \sin nx$ for $n \in \mathbf{N}$. Then, we have

$$(LS^{-1} - I)\sin nx = \frac{p(x)}{n}\cos nx + \frac{q(x)}{n^2}\sin nx.$$

We will later determine the value of N. We set

$$x_r := \sum_{n=N+1}^{\infty} a_n \sin nx \in X_r.$$

Then we have

$$\begin{aligned}
\|(LS^{-1} - I)x_r\| \leq & \|p(x)\sum_{n=N+1}^{\infty} \frac{a_n \cos nx}{n}\| + \|q(x)\sum_{n=N+1}^{\infty} \frac{a_n \sin nx}{n^2}\| \\
\leq & \|p\|_\infty \|\sum_{n=N+1}^{\infty} \frac{a_n \cos nx}{n}\| + \|q\|_\infty \|\sum_{n=N+1}^{\infty} \frac{a_n \sin nx}{n^2}\| \\
\leq & \frac{\|p\|_\infty}{N+1}\|\sum_{n=N+1}^{\infty} a_n \cos nx\| + \frac{\|q\|_\infty}{(N+1)^2}\|\sum_{n=N+1}^{\infty} a_n \sin nx\| \\
= & (\frac{\|p\|_\infty}{N+1} + \frac{\|q\|_\infty}{(N+1)^2})\|\sum_{n=N+1}^{\infty} a_n \sin nx\| \\
= & (\frac{\|p\|_\infty}{N+1} + \frac{\|q\|_\infty}{(N+1)^2})\|x_r\|
\end{aligned}$$

It follows that

$$\|LS^{-1} - I\| \leq \frac{\|p\|_\infty}{N+1} + \frac{\|q\|_\infty}{(N+1)^2}. \tag{3.4}$$

We choose the value of N such that the right-hand side of (3.4) is smaller than $1/2$. This condition satisfies the second inequality in (3.3). Let $N = N_0$ be the smallest natural number satisfying such condition. We determine the value of N $(\geq N_0)$ such that

$$A_N X_N \subset X_N.$$

Here, A_N is the operator defined by

$$A_N \sin nx := \begin{cases} L\sin nx & \text{for} \quad n \leq N_1, \\ S\sin nx = -n^2 \sin nx & \text{for} \quad n > N_1. \end{cases}$$

Since the operator $LA_N^{-1} - I : X_N \to Y$ is expressed by a finite matrix, we can compute the operator norm $\|LA_N^{-1} - I\|$ by the use of computers. We determine N_1 so large that the first inequality of (3.3) holds.

References

[K1] T. Kawanago, Generalization of bifurcation theorems aiming for the study of non-linear forced vibrations described by PDEs, Preprint.

[K2] T. Kawanago, in preparation.

[Ko] Y. Komatsu, A bifurcation phenomenon for the peroodic solutions of a semilinear dissipative wave equation, Preprint.

[M] A. Matsumura, Bifucation phenomena for the Duffing equation, in 'Advances in Nonlinear Partial Differential Equations and Stochastics', Kawashima and Yanagisawa ed. , World Scientific, 1998.

[O] S. Oishi,'Hisenkei kaiseki nyumon ' (in Japanese) (English translation: Introduction to nonlinear analysis), Corona-sha, 1997.

[R] P.H. Rabinowitz, Periodic solutions of non-linear hyperbolic partial differential equations, Comm. Pure. Applied Math. 20 (1967) 145-205.

[WS] B. Werner and A. Spence, The computation of symmetry-breaking bifurcation point, SIAM J. Numer. Anal. 21(1984), p388.